Dynamic Stability Criteria of Nonlinear Elastic Damped/Undamped Systems Under Step Loading

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The nonlinear dynamic stability of simple nonlinearly elastic systems with or without viscous damping under step loading of infinite or finite duration is discussed. The analysis refers to discrete structural systems with one or two degrees of freedom which, under the action of the same loading applied statically, lose their stability by snapping. Considering the stability of motion in the large, in the sense of Lagrange, conditions for a bounded or an unbounded motion and thereby dynamic buckling criteria are properly established for one degree-of-freedom systems. It is found that, regardless of the length of time of loading application as well as regardless of whether or not damping is included, the motion ceases to be bounded when the phase-point velocity vanishes. Using this criterion, exact or approximate (lower bound) dynamic buckling loads for damped or undamped systems under step loading of infinite or finite duration can be obtained without solving the strongly nonlinear differential equation of motion.

Introduction

A LARGE amount of literature has been devoted during the last twenty-five years to the postbuckling analyses of structures. This is due to the fact that by such an analysis one can evaluate the actual load-carrying capacity of structures and hence avoid a possible catastrophic failure associated with an unstable secondary equilibrium path. Contrary to static buckling of structures, which is fairly well-understood, dynamic buckling of autonomous and nonautonomous systems needs further clarifications in view of some lack of pertinent literature. Some early studies[1-4] on dynamic buckling due to step loading (autonomous systems) concern simple structural systems which, under the action of the same loading applied statically, experience snap-through buckling. An interesting extension of Koiter's postbuckling analysis to the dynamic buckling of imperfection sensitive systems under step loading of infinite or finite duration was given in Refs. 5–8. An excellent review on this subject was given by Simitses.[9] Recently, a thorough extension of the foregoing studies to general limit-point systems of one and two degrees of freedom in the particular case under step loading of infinite duration has been presented in Refs. 10–14.

The present work—being a thorough extension of the aforementioned studies—is aiming to give some light regarding the mechanism of dynamic instability of limit-point systems, mainly of one degree of freedom under step loading of infinite or finite duration in which the effect of Rayleigh's dissipative forces is also included. Nonlinear dynamic stability criteria corresponding to all of these cases will be comprehensively discussed. Conditions under which exact (or lower bounds of) dynamic buckling loads can be established by using a static method of analysis will be also examined. The analysis includes both geometrical and material nonlinearities.

General Considerations

We consider holonomic systems of nonlinearly elastic material, described by generalized normal coordinates, which are subjected to a step loading of infinite or finite duration. Attention will be focused primarily on single-degree-of-freedom systems. The corresponding equations of motion, which include material and geometrical nonlinearities, are exact in the sense that not any simplification has been made for its derivation. The Lipschitz condition is assumed to be satisfied and hence the existence and uniqueness of solution of the nonlinear initial-value problem are guaranteed.

Among the three basic definitions of stability attributed to Laplace, Liapunov, and Poincaré, that of Laplace (or Lagrange) is adopted herein. Such stability requiring the boundedness of solution for any time, although too general, can be more conveniently applied to the case of dynamic buckling. Considering the stability of motion in the large, dynamic buckling is defined as that critical state which corresponds to the smallest load for which the motion ceases to be bounded. From a practical viewpoint, dynamic buckling is meant to be that state for which small changes in the magnitude of loading lead to large changes in the system response.

Mathematical Analysis

Let us first consider limit-point systems of one degree of freedom. A typical example of such a system, in which the effect of viscous damping is accounted for, is shown in Fig. 1. The system consists of two rigid links, of equal length l, hinged to each other; a concentrated mass m placed at the hinge is supported by a nonlinear quadratic spring character-

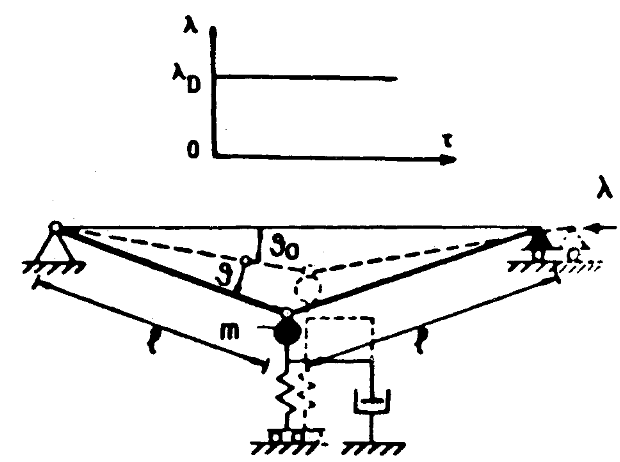

Fig. 1 Single-degree-of-freedom damped system under step loading of infinite duration.

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ized by a linear and nonlinear component k and α, respectively, as well as a viscous damping coefficient c. An initial imperfection is identified with an angle deviation θ_o measured from the straight (unstressed) configuration, while θ is the total angle due to the action of an axial compressive loading suddenly applied. For such a system[7–8] in which the effect of axial inertia of the concentrated mass is also included,[11,12] the equation of motion in dimensionless form is given by

$$\ddot{\theta} + \overset{*}{c}\dot{\theta}\cos^2\theta + [\sin\theta - \sin\theta_o - \alpha(\sin\theta - \sin\theta_o)^2]\cos\theta - \lambda\sin\theta = 0 \tag{1}$$

where $\theta = \theta(\tau) < \pi/2$ (due to physical geometrical restrictions), $\tau = t(k/m)^{1/2}$, $\overset{*}{c} = c/(km)^{1/2}$; t is the physical time coordinate, while λ is the step loading in dimensionless form (i.e., dynamic loading divided by the corresponding static buckling load). Note that the perfect system under static loading exhibits an asymmetric branching point.

A first integral of Eq. (1)—if the system is initially ($\tau = 0$) at rest—yields the total energy equation valid for $\tau \geq 0$:

$$\frac{1}{2}\dot{\theta}^2 + \overset{*}{c}\int_0^\tau \dot{\theta}^2\cos^2\theta \, d\tau' + \frac{1}{2}(\sin\theta - \sin\theta_o)^2 - \frac{\alpha}{3}(\sin\theta - \sin\theta_o)^3 - \lambda(\cos\theta_o - \cos\theta) = 0 \tag{2}$$

From a rigorous dynamic analysis for a single-degree-of-freedom system without damping, it was established[11,12] that the exact dynamic buckling load can be determined using the static stability criterion according to which, at the instant where the motion ceases to be bounded,

$$\dot{\theta} = \ddot{\theta} = 0 \tag{3}$$

or equivalently, the phase-point velocity becomes zero.

Step Loading of Infinite Duration

For a step loading of infinite duration, the conditions of Eq. (3) with the aid of Eqs. (1) and (2), when there is no damping, yield

$$[\sin\theta - \sin\theta_o - \alpha(\sin\theta - \sin\theta_o)^2]\cos\theta - \lambda\sin\theta = 0 \tag{4a}$$

$$\frac{1}{2}(\sin\theta - \sin\theta_o)^2 - \frac{\alpha}{3}(\sin\theta - \sin\theta_o)^3 - \lambda(\cos\theta_o - \cos\theta) = 0 \tag{4b}$$

The solution of Eqs. (4) gives the dynamic buckling load λ_D and the corresponding critical angle θ_D. Note that Eq. (4a) is the static equilibrium equation, whereas Eq. (4b) defines that point on the secondary equilibrium path for which the total potential energy of the system vanishes. Thus, the dynamic critical point is a singular (equilibrium) point. The curve λ vs θ can also be obtained with the aid of numerical integration of Eq. (1) using, for instance, the Runge–Kutta's scheme with a third- or fourth-order accuracy and a step size between 0.05–0.01. In this case, θ is the amplitude of the vibratory motion, i.e., $\theta = \theta_{\max}$. A typical plot showing the static equilibrium path (λ vs θ), as well as the dynamic curve (λ vs $\theta_{\max}$), is shown in Fig. 2. Obviously, $\theta_{\max} = \theta_D$. The length of time, up to the instant of dynamic buckling, is evaluated with the aid of Eq. (2) for $\overset{*}{c} = 0$ as follows:

$$\tau_D = \int_{\theta_o}^{\theta_{\max}} \frac{d\theta}{\sqrt{\frac{2\alpha}{3}(\sin\theta - \sin\theta_o)^3 - (\sin\theta - \sin\theta_o)^2 + 2\lambda(\cos\theta_o - \cos\theta)}} \tag{5}$$

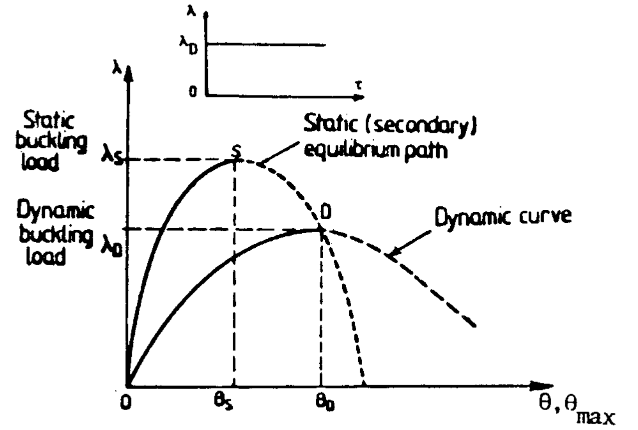

Fig. 2 Typical static (λ vs θ) and dynamic (λ vs $\theta_{\max}$) curves for an imperfect single-degree-of-freedom system.

It is worth observing that the above static stability criterion associated with conditions of Eq. (3) is equivalent to an inflection point in the curve θ vs τ. When the motion is bounded, the maximum of θ occurs at $\dot{\theta} = 0$ with $\ddot{\theta} < 0$. There is no reason for which these observations and the conditions of Eq. (3) do not hold when damping is accounted for.

The solution of Eq. (1)—satisfying the Lipschitz condition—is such that $\theta(\tau) \in C_2$. Thus, a Taylor's expansion of the angular velocity for sufficiently small $\Delta\tau$ gives

$$\dot{\theta}(\tau \pm \Delta\tau) = \dot{\theta}(\tau) \pm \Delta\tau\ddot{\theta}(\tau) \tag{6}$$

Regardless of whether or not the motion is bounded (damped or undamped), the necessary condition for a maximum of θ is $\dot{\theta} = 0$, whereas the sufficient condition is $\ddot{\theta} < 0$. For a loading λ slightly less than the dynamic buckling load λ_D (or λ_{DD} when damping is included), the motion is bounded; the maximum θ occurs for $\dot{\theta}(\tau) = 0$ and hence, a little before and a little after that instant, one can write

$$\dot{\theta}(\tau - \Delta\tau) = -\Delta\tau\ddot{\theta}(\tau) \tag{7a}$$

$$\dot{\theta}(\tau + \Delta\tau) = +\Delta\tau\ddot{\theta}(\tau) \tag{7b}$$

Evidently, $\dot{\theta}(\tau - \Delta\tau)$ and $\dot{\theta}(\tau + \Delta\tau)$ have opposite signs, which means that the motion changes sense at the instant of maximum θ, where $\dot{\theta} = 0$ and $\ddot{\theta} < 0$. Let τ_D (or τ_{DD} if damping is included) be the critical length of time at which the motion ceases to be bounded; then at that time the motion does not change sense and hence sign; that is, $\dot{\theta}(\tau_D - \Delta\tau)$ and $\dot{\theta}(\tau_D + \Delta\tau)$ have the same sign. This, in view of Eqs. (7), yields $\ddot{\theta}(\tau_D) = 0$. Thus, the conditions of Eq. (3) have been rederived in a different way regardless of whether or not damping is included.

Using a Taylor's expansion, it was established[12] that the dynamic buckling load λ_{DD} (when damping is included) is always higher than λ_D, i.e., $\lambda_D < \lambda_{DD}$. The buckling load λ_{DD} can be established either through numerical integration of Eq. (1) or by solving the system of equations associated with the conditions of Eq. (3), i.e.,

$$[\sin\theta - \sin\theta_o - \alpha(\sin\theta - \sin\theta_o)^2]\cos\theta - \lambda\sin\theta = 0 \tag{8a}$$

$$\overset{*}{c}\int_o^\tau \dot{\theta}^2\cos^2\theta \, d\tau' + \frac{1}{2}(\sin\theta - \sin\theta_o)^2 - \frac{\alpha}{3}(\sin\theta - \sin\theta_o)^3 - \lambda(\cos\theta_o - \cos\theta) = 0 \tag{8b}$$

Clearly, the dynamic critical point $(\theta_{DD},\lambda_{DD})$ lies again on the static equilibrium path and more specifically at that equilibrium point for which Eq. (8b) is satisfied. Such a point is located above the corresponding equilibrium point of the undamped system[12] since $\lambda_D < \lambda_{DD}$.

A very good approximation $\tilde{\lambda}_{DD}$ of the latter load can be obtained as follows. Given that the length of time τ_D until the instant of dynamic buckling is very close to $\tau = \tau_{DD}$ (when damping is included), one can write

$$\left(\int_0^{\tau} \dot{\theta} \cos\theta \, d\tau'\right)^2 \le \tau \int_0^{\tau} \dot{\theta}^2 \cos^2\theta \, d\tau' \le \tau_D \int_0^{\tau} \dot{\theta}^2 \cos^2\theta \, d\tau' \quad (9a)$$

or

$$(\sin\theta - \sin\theta_o)^2 \le \tau_D \int_0^{\tau} \dot{\theta}^2 \cos\theta \, d\tau' \quad (9b)$$

By virtue of Eq. (9), Eq. (8b) becomes

$$\left(\frac{1}{2} + \frac{\overset{*}{c}}{\tau_D}\right)(\sin\theta - \sin\theta_o)^2 - \frac{\alpha}{3}(\sin\theta - \sin\theta_o)^3 - \lambda(\cos\theta_o - \cos\theta) \le 0 \quad (10)$$

The solution of Eq. (8a) and Eq. (10) gives the lower bound buckling estimates $\tilde{\lambda}_{DD}$ which are very close to the exact buckling loads λ_{DD}, as we can see from Table 1. Indeed the maximum difference between λ_{DD} and $\tilde{\lambda}_{DD}$ for the extreme case $\overset{*}{c} = 0.10$ and $\theta_o = 0.50$ is less than 2%. Note also that in all cases considered, the critical length of time until dynamic buckling is more than twice the natural period of the system.

From Fig. 3, one can see the static equilibrium path as well as the relationship between λ and θ_{max} (amplitude of vibration) for an undamped ($\overset{*}{c} = 0$) and a damped ($\overset{*}{c} = 0.10$) system with $\theta_o = 0.05$. In this figure, the curve λ vs $\theta(=\theta_{max})$ obtained from the solution of Eqs. (8a) and Eq. (10) is also plotted with a dotted line (being very close to the corresponding exact curve). Additional results for a damped system with $\overset{*}{c} = 0.10$ and $\theta_o = 0.05$ are given in Figs. 4a and 4b. More specifically, bounded ($\lambda < \lambda_{DD}$) and unbounded ($\lambda > \lambda_{DD}$) motions in the phase plane $(\dot{\theta}, \theta)$ are shown in Fig. 4a, while Fig. 4b shows respective variations of the angular velocity with respect to time ($\dot{\theta}$ vs τ). The corresponding critical quantities are $\lambda_D = 0.49778$ and $\theta_D = 0.22170$.

Step Loading of Finite Duration

For a step loading with time duration $\bar{\tau} < \tau_D$ (or τ_{DD} when damping is included), the equation of motion for $\tau > \bar{\tau}$ is given in relation (1). In this case, dynamic buckling is anticipated to occur at a buckling load λ_D (or λ_{DD} when damping is included) higher than that corresponding to a step loading of infinite duration. Moreover, dynamic buckling is expected to take place at a time τ_D (or τ_{DD}) greater than (or equal to) $\bar{\tau}$. In such a case, the corresponding equation and its first integral are given by

$$\ddot{\theta} + \overset{*}{c}\dot{\theta} \cos^2\theta + [\sin\theta - \sin\theta_o - \alpha(\sin\theta - \sin\theta_o)^2] \cos\theta = 0 \quad (11a)$$

$$\frac{1}{2}\dot{\theta}^2 + \overset{*}{c} \int_{\bar{\tau}}^{\tau} \dot{\theta}^2 \cos^2\theta \, d\tau' + \frac{1}{2}(\sin\theta - \sin\theta_o)^2 - \frac{\alpha}{3}(\sin\theta - \sin\theta_o)^3 - \lambda(\cos\theta_o - \cos\bar{\theta}) = 0 \quad (11b)$$

where $\bar{\theta}$ corresponds to $\tau = \bar{\tau}$ and is evaluated by step increasing the loading with the aid of Eq. (1) subject to the initial conditions $\theta(0) = \theta_o$ and $\dot{\theta}(0) = 0$.

Application of the dynamic buckling criterion associated with an inflection point in the curve θ vs τ, i.e., $\dot{\theta} = \ddot{\theta} = 0$, leads to the following equations resulting from Eq. (11):

$$[\sin\theta - \sin\theta_o - \alpha(\sin\theta - \sin\theta_o)^2] \cos\theta = 0 \quad (12a)$$

$$\overset{*}{c} \int_{\bar{\tau}}^{\tau} \dot{\theta}^2 \cos^2\theta \, d\tau' + \frac{1}{2}(\sin\theta - \sin\theta_o)^2 - \frac{\alpha}{3}(\sin\theta - \sin\theta_o)^3 - \lambda(\cos\theta_o - \cos\bar{\theta}) = 0, \quad \text{for } \tau \ge \bar{\tau} \quad (12b)$$

Since Eq. (12a) is not an equilibrium equation, the dynamic critical point is no longer a singular (equilibrium) point. This

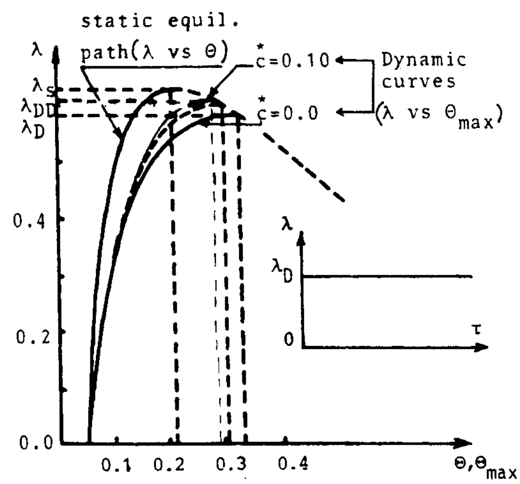

Fig. 3 Static and dynamic curves of a single-degree-of-freedom imperfect ($\theta_o = 0.05$) system with and without damping.

Table 1 Exact and approximate dynamic buckling loads λ_{DD} and $\tilde{\lambda}_{DD}$ for various values of θ_o and $\overset{*}{c}$ of a system with $\alpha = 1$ under step load of infinite duration

Initial imperfection θ_o	Damping coefficient $\overset{*}{c}$	Dynamic buckling load λ_{DD}	$\tilde{\lambda}_{DD}$	τ_D (τ_D/τ_o)	Static buckling load λ_s
0.01	0.00	0.78903			
	0.06	0.79746	0.79422	17.55	0.81502
	0.10	0.80218	0.79792	(2.974)	
0.05	0.00	0.58235			
	0.06	0.59226	0.58769	15.446	0.62589
	0.10	0.59821	0.59104	(2.458)	
0.10	0.00	0.46121			
	0.06	0.47057	0.46589	14.820	0.50919
	0.10	0.47629	0.46887	(2.358)	
0.50	0.00	0.14742			
	0.06	0.15071	0.14912	13.567	0.17610
	0.10	0.15282	0.15023	(2.159)	

Note: τ_o is the natural period of the unloaded system, i.e., $\tau_o = 2\pi\sqrt{m/k}$.

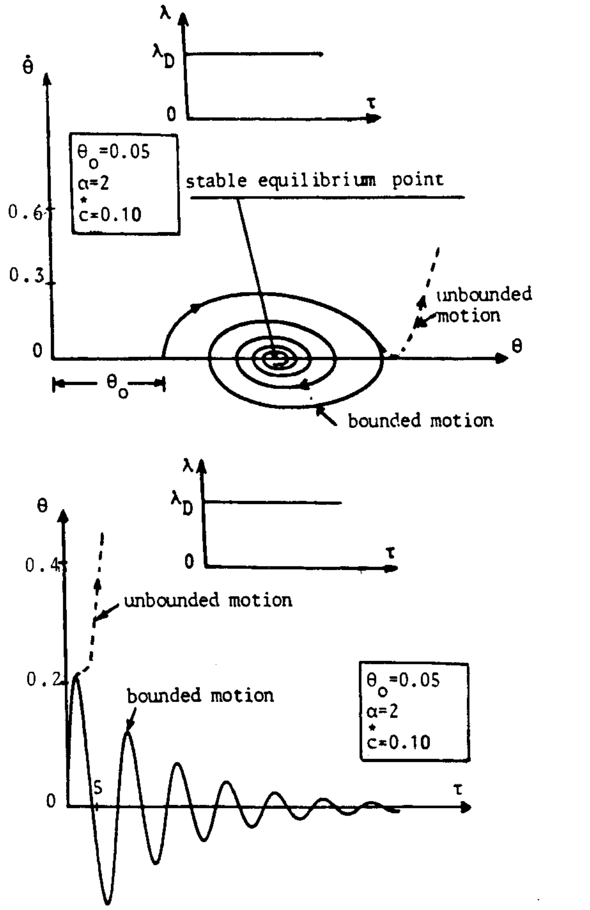

Fig. 4 Phase-plane ($\dot\theta-\theta$) motion (a) and angular velocity $\dot\theta$ vs time τ (b) of a single-degree-of-freedom system.

equation for $\theta < \pi/2$ implies

$$\sin\theta - \sin\theta_o = \frac{1}{\alpha} \tag{13}$$

since the case $\sin\theta - \sin\theta_o = 0$ must be excluded because the angle θ takes its maximum value at the instant of dynamic buckling and hence $\theta > \theta_o$. In view of Eq. (13), dynamic buckling is possible for $\theta < \pi/2$ only when

$$\alpha \geq \frac{1}{1-\sin\theta_o}. \tag{14}$$

Substituting Eq. (13) into Eq. (12b) yields

$$\overset{*}{c}\int_{\bar\tau}^{\tau} \dot\theta^2 \cos^2\theta \, \mathrm{d}\tau' + \frac{1}{6\alpha^2} - \lambda(\cos\theta_o - \cos\bar\theta) = 0 \tag{15}$$

If there is no damping, the dynamic buckling load is given by

$$\lambda_D = \frac{1}{6\alpha^2(\cos\theta_o - \cos\bar\theta)} \tag{16}$$

where $\bar\theta$ and λ_D must satisfy the resulting from relation (5) equation

$$\bar\tau = \int_{\theta_o}^{\bar\theta} \frac{\mathrm{d}\theta}{\sqrt{\frac{2\alpha}{3}(\sin\theta - \sin\theta_o)^3 - (\sin\theta - \sin\theta_o)^2 + 2\lambda_D(\cos\theta_o - \cos\theta)}} \tag{17a}$$

which due to Eq. (16) yields

$$\bar\tau = \int_{\theta_o}^{\bar\theta} \frac{\mathrm{d}\theta}{\sqrt{\frac{2\alpha}{3}(\sin\theta - \sin\theta_o)^3 - (\sin\theta - \sin\theta_o)^2 + \frac{1}{3\alpha^2}\frac{(\cos\theta_o - \cos\theta)}{(\cos\theta_o - \cos\bar\theta)}}} \tag{17b}$$

When damping is accounted for, an exact dynamic buckling load λ_{DD} can be obtained only by means of numerical integration of Eq. (11a), which at the instant of buckling must satisfy the condition of Eq. (15). However, without using such a procedure one could obtain a very good approximation of the dynamic buckling load λ_{DD} as follows: Given that

$$\left(\int_{\bar\tau}^{\tau} \dot\theta \cos\theta \, \mathrm{d}\tau'\right)^2 \leq (\tau_D - \bar\tau)\int_{\bar\tau}^{\tau} \dot\theta^2 \cos^2\theta \, \mathrm{d}\tau' \tag{18}$$

the condition of Eq. (15) can be replaced by the inequality

$$\frac{\overset{*}{c}}{(\tau_D - \bar\tau)}(\sin\theta - \sin\bar\theta)^2 + \frac{1}{6\alpha^2} - \lambda(\cos\theta_o - \cos\bar\theta) \leq 0 \tag{19}$$

where $\lambda = \lambda_{DD}$, while a good approximation of $\bar\theta$ can be obtained from Eq. (17b); τ_D is evaluated from Eq. (5) in which $\theta_{\max} = \theta_D$ and $\lambda = \lambda_D$ correspond to the solution of Eqs. (4). Relation (19) by virtue of relations (13) and (18) yields

$$\lambda_{DD} \geq \frac{1}{(\cos\theta_o - \cos\bar\theta)}\left[\frac{1}{6\alpha^2} + \frac{\overset{*}{c}}{(\tau_D - \bar\tau)} \times \left(\frac{1}{\alpha} + \sin\theta_o - \sin\bar\theta\right)^2\right] = \bar\lambda_{DD} \tag{20}$$

Clearly, $\bar\lambda_{DD}$ ($> \lambda_D$) is a lower-bound dynamic buckling estimate better than λ_D [obtained from Eq. (16)].

Numerical results for a step loading of finite duration corresponding to a system having $\alpha = 2$, $\theta_o = 0.05$ with $\overset{*}{c} = 0$ and $\overset{*}{c} = 0.10$ are given below in Figs. 5–12.

From Figs. 5a and 5b, one can see the variations of θ and $\dot\theta$ vs τ as well as the variations of θ and $\ddot\theta$ vs τ for the cases of a bounded ($\lambda < \lambda_D$) and an unbounded ($\lambda > \lambda_D$) motion. The maximum θ, i.e., $\theta_{\max}$ (occurring at $\tau_D = 14.89$ and $\lambda_D = 0.4937$) corresponds to $\dot\theta = \ddot\theta = 0$. Figure 6 shows the variation the phase-point velocity $(\dot\theta^2 + \ddot\theta^2)^{1/2}$ vs τ, whereas Fig. 7 depicts the response of the system in the phase-plane ($\dot\theta$,θ). Similar results are shown in Figs. 8a, 8b, 9, and 10 when damping ($\overset{*}{c} = 0.10$) is accounted for.

The variation of λ vs the amplitude of vibration $\theta_{\max}$ up to the dynamic buckling load λ_D (or λ_{DD}) associated with θ_D (or θ_{DD}) is displayed in Fig. 11. From this figure, it is clear that λ_D (or λ_{DD}) corresponds to a limit point (maximum) in the curve λ vs $\theta_{\max}$. Finally, from Fig. 12, we can see the variation of λ_D (and λ_{DD}) with respect to the length of time τ_o (of application of the rectangular load), where $\tau_o = \tau/2\pi\sqrt{m/k}$. Note that the dynamic buckling load λ_D (or λ_{DD}) corresponding to a length of time τ can be viewed as dynamic buckling due to the finite time impulse $I = \lambda_D \tau$ for rectangular loading.

From Table 2, evaluated for $\alpha = 2$, we can see the variation of the dynamic buckling load λ_D (or λ_{DD}) for a model with various values of θ_o (=0.01,0.05,0.10,0.50) and $\overset{*}{c}$ (=0,0.10) which is subjected to various lengths of time $\bar\tau$ (=5 and 10). The values between parentheses correspond to dynamic buckling loads for step loadings of infinite duration. Note also that τ_{DD} is always slightly higher than τ_D.

In view of the above analysis, it is not difficult to show that the inflection-point dynamic instability criterion holds also for other types of forcing functions.

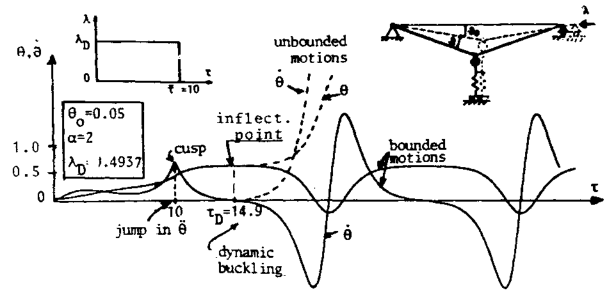

Fig. 5a Variations of θ and $\dot{\theta}$ vs τ for an undamped single-degree-of-freedom system under step loading of finite duration.

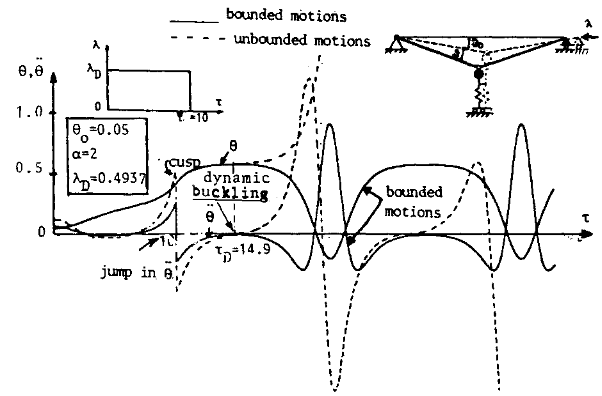

Fig. 5b Variations of θ and $\ddot{\theta}$ vs τ for an undamped single-degree-of-freedom system under step loading of finite duration.

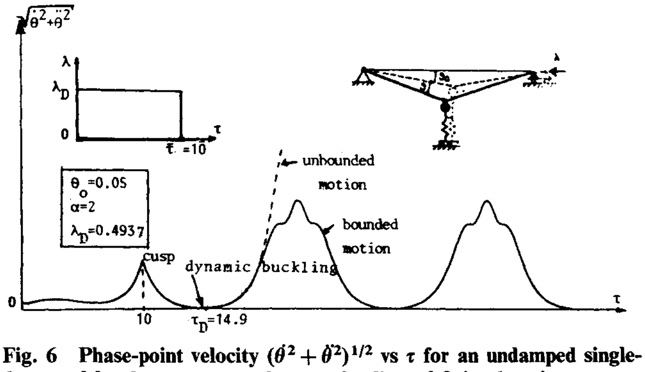

Fig. 6 Phase-point velocity $(\dot{\theta}^2+\ddot{\theta}^2)^{1/2}$ vs τ for an undamped single-degree-of-freedom system under step loading of finite duration.

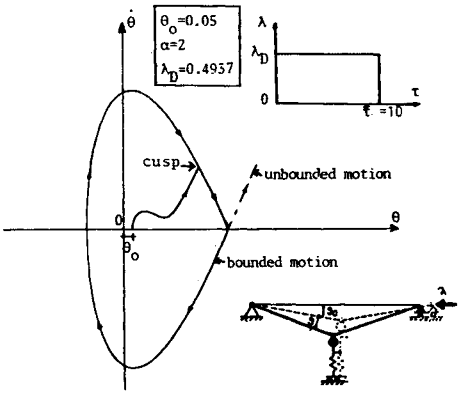

Fig. 7 The phase-plane $(\dot{\theta},\theta)$ motions of an undamped single-degree-of-freedom system under step loading of finite duration.

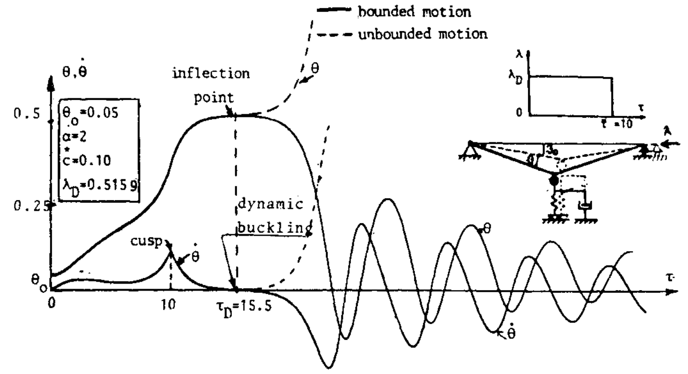

Fig. 8a Variations of θ and $\dot{\theta}$ vs τ for a damped single-degree-of-freedom system under step loading of finite duration.

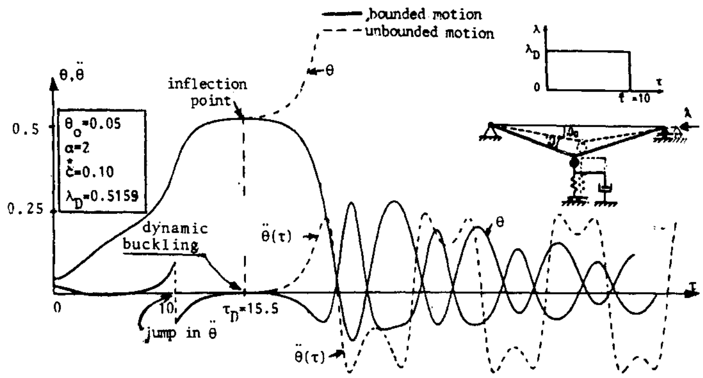

Fig. 8b Variations of θ and $\ddot{\theta}$ vs τ for a damped single-degree-of-freedom system under step loading of finite duration.

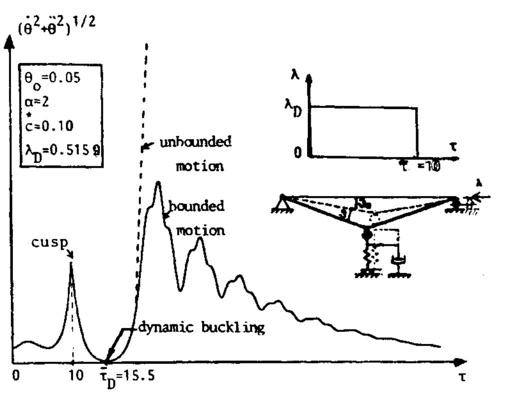

Fig. 9 Phase-point velocity $(\dot{\theta}^2+\ddot{\theta}^2)^{1/2}$ vs τ for a damped single-degree-of-freedom system under step loading of finite duration.

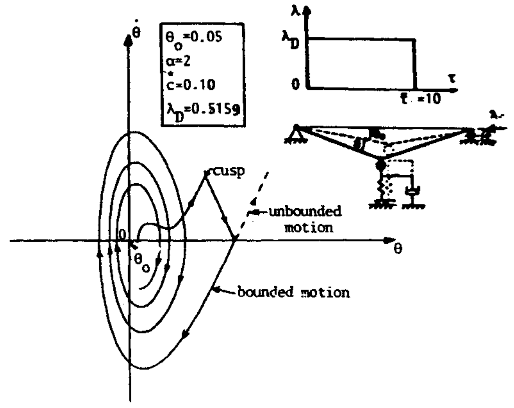

Fig. 10 Bounded and unbounded motion in the phase plane $(\dot{\theta},\theta)$ of a damped single-degree-of-freedom system under step loading of finite duration.

Table 2 Critical loads λ_D (or λ_{pp}) and critical angles θ_D and θ_{DD} of a system under step loading of finite duration $\bar{\tau}$ for $\alpha = 2$ and various values of $\overset{*}{c}$ and θ_o

Initial imperfection θ_o	Damping coefficient $\overset{*}{c}$	$\bar{\tau}$	$\bar{\theta}$	Buckling time length τ_D or τ_{DD}	Buckling quantities λ_D or λ_{DD}	θ_D or θ_{DD}
0.01	0.00	10	0.32994	16.52	0.77328	0.53365
		∞		(18.91)	(0.72008)	(0.11247)
	0.10	10	0.33581	15.12	0.80777	0.53075
		∞		(21.42)	(0.73468)	(0.09994)
0.05	0.00	10	0.41702	14.89	0.49369	0.57647
		∞		(14.75)	(0.48193)	(0.23737)
	0.10	10	0.42020	14.52	0.51587	0.57555
		∞		(15.39)	(0.49778)	(0.22170)
0.10	0.00	10	0.49294	16.16	0.36571	0.64104
		∞		(14.76)	(0.35990)	(0.33129)
	0.10	10	0.49462	15.04	0.38326	0.63854
		∞		(14.73)	(0.37408)	(0.31456)
0.50	0.00	10	1.04200	23.36	0.11172	1.33360
		∞		(15.52)	(0.10827)	(0.85328)
	0.10	10	1.03984	25.10	0.11735	1.35041
		∞		(16.31)	(0.11291)	(0.83528)
0.01	0.00	5	0.26973	11.36	1.14826	0.53268
	0.10	5	0.27332	13.60	1.22239	0.53600
0.05	0.00	5	0.36520	12.30	0.63181	0.58171
	0.10	5	0.37042	12.42	0.67514	0.58243
0.10	0.00	5	0.43512	10.90	0.44715	0.63958
	0.10	5	0.44801	11.26	0.47881	0.64102
0.50	0.00	5	0.94182	22.40	0.14099	1.35461
	0.10	5	0.94964	23.78	0.15013	1.35813

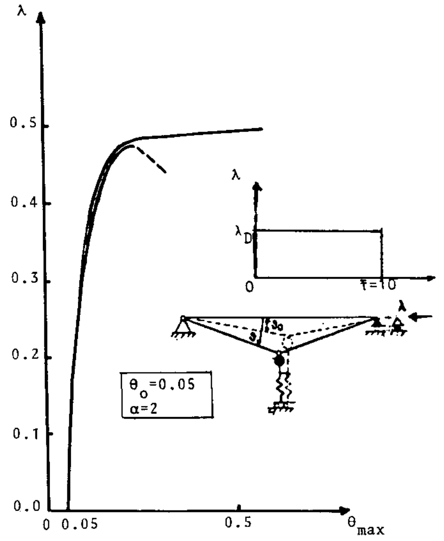

Fig. 11 Variation of λ vs θ_{max} for a damped ($\overset{*}{c} = 0.10$) and an undamped ($\overset{*}{c} = 0$) single-degree-of-freedom system under step loading of finite duration.

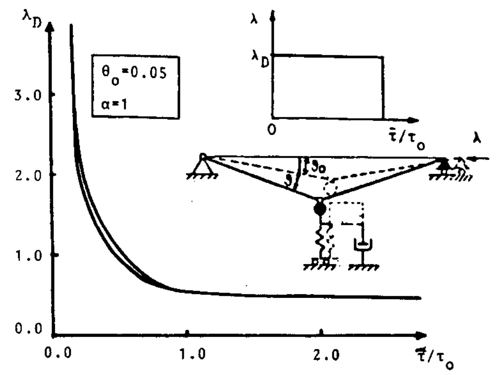

Fig. 12 Dynamic buckling loads λ_D and λ_{DD} vs time length τ/τ_o for a damped ($\overset{*}{c} = 0.10$) and an undamped ($\overset{*}{c} = 0$) single-degree-of-freedom system under step loading of varying duration.

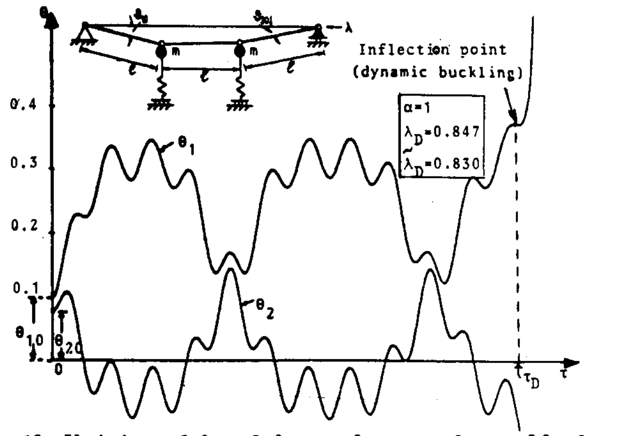

Fig. 13 Variations of θ_1 and θ_2 vs τ for a two-degree-of-freedom undamped imperfect ($\theta_{10} = 0.10$ and $\theta_{20} = 0.08$) system under step loading of infinite duration.

Two-Degree-of-Freedom System

The inflection-point instability criterion can be extended to systems with more than one degree of freedom. In this case, dynamic buckling occurs when the motion corresponding to at least one of its displacement components becomes unbounded.[13,14] For instance, such a state for the two-degree-of-freedom system shown in Fig. 13 occurs when $\dot{\theta}_1 = \ddot{\theta}_1 = 0$. The variation of both angles vs τ up to the instant of dynamic buckling for an undamped system of two degrees of freedom with $\alpha = 1$ and initial angle deviations $\theta_{10} = 0.10$ and $\theta_{20} = 0.08$ is shown in Fig. 13. The dynamic buckling load in this case is equal to $\lambda_D = 0.847$,[13] whereas the approximate buckling load associated with zero total potential energy is $\tilde{\lambda}_D = 0.830$.

Conclusions

The most important conclusions for one-degree-of-freedom limit-point damped or undamped systems under step loading of infinite or finite duration are the following:

1) Regardless of whether or not damping is taken into account, dynamic buckling (associated with an unbounded motion) occurs at the smaller value of the loading for which the phase-point velocity vanishes. This stability criterion is

equivalent to the existence of an inflection point on the curve "displacement vs time." When the step loading is of infinite duration, the "dynamic critical point" coincides with an unstable equilibrium point (with zero total potential energy if damping is ignored), whereas in the case of a step loading of finite duration, the "dynamic critical point" is a regular (nonsingular) point.

2) Application of the static stability criterion associated with zero total potential energy allows us to determine the exact dynamic buckling loads when damping is ignored and lower-bound buckling estimates if this effect is included. A better estimate very close to the exact dynamic buckling load is also established.

3) The dynamic buckling load for a step loading of infinite duration is always less than the static buckling load (being an upper bound of the former buckling load).

4) The difference between dynamic and static buckling load increases considerably as the initial angle imperfection increases. The imperfection sensitivity under dynamic load is stronger than that under the same load applied statically.

5) The dynamic buckling load in case of a step loading of finite duration is much higher than that of a step loading of infinite duration. More specifically, such a buckling load increases considerably as the length of time application of the loading increases; moreover, it is much higher than the corresponding static buckling load.

6) dynamic buckling estimates for damped systems under step loading of rectangular shape are also established.

7) Viscous damping always causes an increase in the dynamic buckling load. Such an effect increases with the increase of the initial imperfection.

8) The inflection-point dynamic instability criterion holds for step loads with various forcing functions.

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